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Sato's Tau-functions expressed by Weyl-functions and its application to KdV flow

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1 Introduction

[GGKM] discovered that Schrödinger operators with potentials of solutions to the KdV equation

$$\partial_t f = 6f\partial_x f - \partial_x^3 f$$

are unitarily equivalent, and became a trigger for a rapid development of completely integrable non-linear partial differential equations. Since then, most of the works have been done by using the scattering data of associated Schrödinger operators for decaying solutions, or by using the discriminant for periodic solutions, which has restricted ourselves to consider mostly the two classes of solutions to the KdV equation, namely decaying or periodic ones. The best results in these two categories are as follows. In [CKSTT] they showed the KdV equation is uniquely solvable in the Sobolev space $H^s(\boldsymbol{R})$ with $s > -3/4$, and in [KT] they obtained the global wellposedness in $H^{-1}(\boldsymbol{T}, \boldsymbol{R})$.

There are several works treating solutions which are not decaying nor periodic. [Eg] was the first in which she studied almost periodic solutions to the KdV equation, however the class she considered was a certain class of limit periodic solutions. Analytic quasi periodic solutions were treated in [DG] and [BDGL], although their class of initial data had to be very small. A general quasi periodic solutions was studied in [Tsu], however, the global wellposedness has not been shown. Step-like solutions decaying on the right half axis have been investigated by [Ryb2] and [GR]. His method is interesting from our point of view, since he uses the Hirota's tau-functions. However, it has an objection that the scattering data are used in the definition of the tau-function, which seems to prevent him from extending the class of solutions.

On the other hand, the algebraic structure of the KdV equation was revealed by [Sat] and yielded a unified approach to a wide class of integrable systems. Since his argument was algebraic, so obtained solutions were rational, multi-solitons and algebro-geometric ones, although all these solutions were described by Tau-functions in a unified way. It has been a problem to what extent this method is effective to obtain general solutions to the KdV equation such as solutions starting from almost periodic functions. [SW] considered a kind of closure of Sato's framework to obtain a certain class of transcendental solutions. However, their solutions still remain in a meromorphic class on the entire complex plane $\mathbb{C}$. It should be noted that [Mar] proposed an algorithm

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to construct solutions to the KdV equation, although it seems that his method also has difficulty to go beyond the class investigated by [SW].

In [Ko2] we gave a representation of the Tau-functions by the Weyl functions of Schrödinger operators. Since the Weyl function is quantity defined for general potentials, there is a hope for this representation to give general solutions to the KdV equation. In this paper we give a brief sketch of the proof of the construction of a KdV flow on a space containing the Schwartz space $\mathcal{S}(\boldsymbol{R})$ and smooth almost periodic functions.

To state the results we prepare several notions from spectral theory of Schrödinger operators. For a real valued $q \in L^1(\boldsymbol{R})$ assume that the associated Schrödinger operator L_q

$$L_q f \triangleq -\partial_x^2 f + qf \tag{1}$$

is essentially self-adjoint, which is equivalent to the unique existence of non-trivial solutions $f_\pm$ to $L_q f = zf$ with $f_\pm \in L^2(\boldsymbol{R}_\pm)$ and $f_\pm(0) = 1$ for $z \in \boldsymbol{C}\backslash\boldsymbol{R}$. Its *Weyl functions* $m_\pm$ are defined by

$$m_\pm(z) = \pm\frac{f'_\pm(0,z)}{f_\pm(0,z)}.$$

$m_\pm$ are holomorphic on $\boldsymbol{C}\backslash\boldsymbol{R}$ and have positive imaginary parts (such a function is called Herglotz function). the inverse spectral theory implies that $m_\pm$ uniquely recover a potential q. A potential q is called *reflectionless* on $F \in \mathcal{B}(\boldsymbol{R})$ if its Weyl functions $m_\pm$ satisfy

$$m_+(\xi + i0) = -\overline{m_-(\xi + i0)} \quad \text{a.e. } \xi \in F. \tag{2}$$

Set

$$m(z) = \begin{cases} -m_+\left(-z^2\right) & \text{if } \operatorname{Re} z > 0 \\ m_-\left(-z^2\right) & \text{if } \operatorname{Re} z < 0 \end{cases}, \tag{3}$$

and assume that there exist $\lambda_0 < 0 < \lambda_1$ such that

$$\inf \mathrm{sp} L_q > \lambda_0, \text{ and } q \text{ is reflectionless on } (\lambda_1, \infty).$$

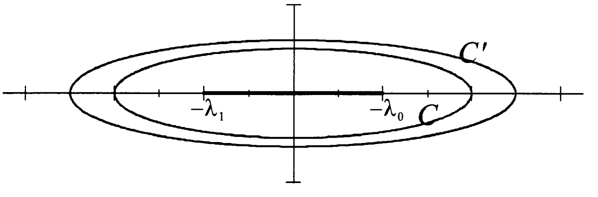
Then, m is holomorphic on $\boldsymbol{C}\backslash\left(\left[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}\right] \cup i\left[-\sqrt{\lambda_1}, \sqrt{\lambda_1}\right]\right)$, and m has a simple pole at ∞ such that

$$m(z) = z + m_1 z^{-1} + m_2 z^{-2} + \cdots$$

holds. Set

$$\begin{cases} \mathcal{Q}_{\lambda_0,\lambda_1} = \{q;\ \inf \mathrm{sp} L_q > \lambda_0 \text{ and } q \text{ is reflectionless on } (\lambda_1, \infty)\} \\ \Gamma = \left\{g;\ g = e^h \text{ with odd polynomial } h \text{ satisfying } h(z) = \overline{h(\overline{z})}\right\} \end{cases}$$

Let C, C' be simple closed curves surrounding the interval $[-\lambda_1, -\lambda_0]$ counter-clockwise. The figure f.1 indicates the situation.



f.1

For a function f denote by f_e, f_o the even part and odd part respectively, namely

$$f_e(z) = \frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}, \quad f_o(z) = \frac{f(\sqrt{z}) - f(-\sqrt{z})}{2\sqrt{z}}.$$

For δ whose δ_e, δ_o are holomorphic in a simply connected domain including C', set $\widetilde{m}(z) = m(z) - \delta(z)$, and define

$$\begin{cases} M_g(z,\lambda) = \dfrac{\widehat{g}_o(z)\,(g\widetilde{m})_e(\lambda) + \widehat{g}_e(z)\,(g\widetilde{m})_o(\lambda)}{\lambda - z} \\ N_g(z,\lambda) = \dfrac{1}{2\pi i}\displaystyle\int_{C'} \dfrac{M_g(\lambda',\lambda)}{\lambda' - z} m_o(\lambda')^{-1}\, d\lambda' \\ (N_m(g)f)(z) = \dfrac{1}{2\pi i}\displaystyle\int_C N_g(z,\lambda)\, f(\lambda)\, d\lambda. \end{cases} \tag{4}$$

for $g \in \Gamma$ and $f \in L^2(C)$, where $\widehat{g}(z) = g(z)^{-1}$. Then, $N_m(g)$ defines a trace class operator on $L^2(C)$. In [Ko2] we announced the following

Theorem 1 *For $q \in \mathcal{Q}_{\lambda_0,\lambda_1}$ and $g \in \Gamma$ define a tau-function by*

$$\tau_m(g) = \det(I + N_m(g)).$$

Then, $\tau_m(g) > 0$ is always valid, and

$$(K(g)q)(x) = -2\partial_x^2 \log \tau_m(ge_x) \quad \textit{with} \quad e_x(z) = e^{xz}$$

defines a smooth flow on $\mathcal{Q}_{\lambda_0,\lambda_1}$ such that

$$\begin{cases} \left(K\left(e^{tz}\right)q\right)(x) = q(x+t), \\ \left(K\left(e^{-4tz^3}\right)q\right)(x) \text{ satisfies the KdV equation.} \end{cases}$$

The class $\mathcal{Q}_{\lambda_0,\lambda_1}$ contains multi-solitons, algebro-geometric solutions and they are dense in $\mathcal{Q}_{\lambda_0,\lambda_1}$. Since $\tau_W(e_x)$ is entire as a function of x, $q(x)$ turns to be meromorphic on the entire complex plane $\boldsymbol{C}$.

2 Main result

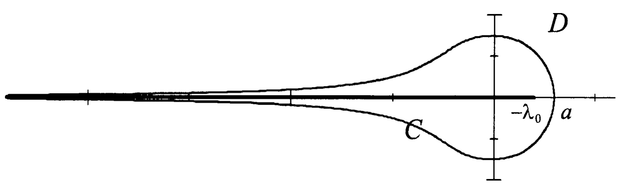
Theorem 1 suggests the possibility to define τ_m for general Weyl functions having no analyticity in a vicinity of ∞. We assume for simplicity $\inf \mathrm{sp} L_q > -\infty$ and fix $\lambda_0 < \inf \mathrm{sp} L_q$. If q has no reflectionless property on any right half axis, we have to extend the curves C, C' to $-\infty$. In order to have $N_m(g)$ as a bounded operator g_e, g_o should remain bounded along C, C'. Suppose $\Gamma \ni g = e^h$ and $h(z) = h_1 z^n +$ lower order term with odd n, and for $a > -\lambda_0$ let $\{\omega(x)\}_{x \le a}$ be a continuous function satisfying

$$\begin{cases} \omega(x) > 0 \text{ on } (-\infty, a) \\ \omega(x) = 0 \text{ on } [a,\infty) \\ \omega(x) = (-x)^{-\alpha} \text{ for } x \le -1 \end{cases}.$$

We assume the curve C is symmetric with respect to the real line, namely $C = \overline{C}$, and C is parametrized by ω by

$$C \text{ on } \boldsymbol{C}_+ = \{x + i\omega(x);\ \ x \le a\}.$$

The shape of C is illustrated in f.2.



f.2

D denotes the outside region of C, that is

$$D = \{z \in \boldsymbol{C}; \quad |\mathrm{Im}\, z| \geq \omega(\mathrm{Re}\, z)\}.$$

Since

$$h\left(\sqrt{z}\right) = h_1 z^{n/2} + \text{lower order term},$$

and for $\alpha > -1$

$$\left(x + i\,(-x)^{-\alpha}\right)^{n/2} = i\,(-x)^{n/2}\left(1 - \frac{n}{2} i\,(-x)^{-\alpha-1} + O\,(-x)^{-2(\alpha+1)}\right)$$

holds as $x \to -\infty$, $e^{h(\pm\sqrt{z})}$ remain bounded as $\mathrm{Re}\, z \to -\infty$ along C if $n/2 - \alpha - 1 \leq 0$, hence so do g_e and g_o.

Keeping this situation in mind, we define a metric

$$d_{\alpha,\beta}\,(m_1, m_2) = \sup_{z \in D} |z|^{\beta}\left(|m_{1,e}(z) - m_{2,e}(z)| + |m_{1,o}(z) - m_{2,o}(z)|\right)$$

for $\alpha > -1$, $\beta > 0$. We denote this C by C_α and let $\mathcal{Q}$ be a set of all potentials whose Weyl functions $m_\pm$ satisfy

$$\left\{\begin{array}{l} m_+(z) = \sqrt{-z} + \sum_{m=1}^{n} a_m \left(\sqrt{-z}\right)^{-m} + o\left(\left(\sqrt{-z}\right)^{-n}\right) \\ m_-(z) = \sqrt{-z} + \sum_{m=1}^{n} b_m \left(\sqrt{-z}\right)^{-m} + o\left(\left(\sqrt{-z}\right)^{-n}\right) \end{array}\right. \tag{5}$$

for any $n \geq 1$ along the curve C_α for any $\alpha > 0$ with real constants $\{a_m, b_m\}$ satisfying

$$a_m = b_m \quad \text{for odd } m, \quad \text{and} \quad a_m = -b_m \quad \text{for even } m.$$

It should be noted that the property (5) holds on any sector $\{\epsilon < \mathrm{Im}\, z < \pi - \epsilon\}$ for any $\epsilon > 0$ if q is smooth at $x = 0$. Clearly $\mathcal{Q}_{\lambda_0,\lambda_1} \subset \mathcal{Q}$ is valid. The first key lemma is

Lemma 1 *(i) $\mathcal{Q}_{\lambda_0,\lambda_1}$ is dense in $\mathcal{Q}$ with metric $d_{\alpha,\beta}$ for any α, $\beta > 0$.*
(ii) $\tau_m\,(g)$ is continuous in m with respect to $d_{\alpha,\beta}$ for every sufficiently large α, β if g is fixed.

Proof. Suppose $m_\pm$ are given by

$$m_\pm\,(z) = i\sqrt{z} + \int_{-\infty}^{\infty} \frac{1}{\xi - z}\left(\sigma_\pm\,(d\xi) - \frac{\sqrt{\xi}_+}{\pi} d\xi\right),$$

and for $r > 1$ set

$$m_\pm^r\,(z) = i\sqrt{z} + \int_r^{\infty} \frac{\rho\,(\xi)}{\xi - z} d\xi + \int_{-r}^{r-1} \frac{1}{\xi - z}\left(\sigma_\pm\,(d\xi) - \frac{1}{\pi}\sqrt{\xi}_+ d\xi\right),$$

where

$$\rho(\xi) = \frac{\sqrt{\xi - r}}{2\pi} \int_{-r}^{r-1} \frac{1}{\xi - \xi'} \frac{\sigma_+ (d\xi') + \sigma_- (d\xi') - \frac{2}{\pi}\sqrt{\xi'}_+ d\xi'}{\sqrt{r - \xi'}}.$$

Then, the associated q_r is reflectionless on (r, ∞). Under the condition on q one can show $m^r \to m$ as $r \to \infty$. In the process of the proof we use a conformal map ϕ from $\boldsymbol{C}_+$ onto $D \cap \boldsymbol{C}_+$. ■

This lemma makes it possible to extend the definition of $\tau_m(g)$ to m such that the associated q is an element of $\mathcal{Q}$, and we have

Theorem 2 *The extended $\tau_m(g)$ satisfies $\tau_m(g) > 0$ and*

$$(K(g)q)(x) = -2\partial_x^2 \log \tau_m(ge_x) \quad \textit{with} \quad e_x(z) = e^{xz}$$

defines a flow on $\mathcal{Q}$, namely $K(g_1 g_2) = K(g_1)K(g_2)$ holds for any g_1, $g_2 \in \Gamma$. If we choose $g_t(z) = e^{tz}$, then $(K(g_t)q)(x) = q(t+x)$, and for $g_t(z) = e^{-4tz^3}$, $u(t,x) = (K(g_t)q)(x)$ yields a solution to the KdV equation.

The definition of $\mathcal{Q}$ is indirect, so the next task is to give simple sufficient conditions for q to be an element of $\mathcal{Q}$.

3 Sufficient conditions

We have to find a rich family of potentials q satisfying (5). The first example is a potential of the Schwartz space $\mathcal{S}(\boldsymbol{R})$. In this case one can use the Jost solutions to estimate $m_\pm$ and without difficulty we have $\mathcal{S}(\boldsymbol{R}) \subset \mathcal{Q}$. The second example is an almost periodic potential. We consider here a much wider class of ergodic potentials. To examine (5) we introduce the other two Herglotz functions

$$m_1(z) = -\frac{1}{m_+(z) + m_-(z)}, \quad m_2(z) = \frac{m_+(z)m_-(z)}{m_+(z) + m_-(z)}.$$

Observe that the condition (5) on $m_\pm$ is equivalent a similar condition on $m_{1,2}$, and that condition is achieved when $\xi_j(z) = (\arg m_j(z))/\pi \in [0,1]$ satisfy

$$\int_0^\infty \lambda^n \left| \xi_j(\lambda) - \frac{1}{2} \right| d\lambda < \infty \tag{6}$$

for any $n \geq 1$ and $j = 1, 2$. The function $\xi_1(\lambda)$ was used by [GS1] and [GS2] to investigate the inverse spectral problem. To examine the condition (6) we need another quantity called *reflection coefficient* defined by

$$R(z) = \frac{m_+(z) + \overline{m_-(z)}}{m_+(z) + m_-(z)},$$

which satisfies $|R(z)| \leq 1$. This quantity was introduced by [Ryb1] and studied by [Rem].

Lemma 2 *Suppose $m_{\pm} \in \boldsymbol{C}_+$ and define $m_{1,2}$ as above. Let $\xi_j = (\arg m_j)/\pi$ and $R = (m_+ + \overline{m_-})/(m_+ + m_-)$. Then, the inequalities below are valid.*

$$\left|\xi_1 - \frac{1}{2}\right|, \quad \left|\xi_2 - \frac{1}{2}\right| \le \frac{2}{\pi}|R|.$$

Therefore, (6) is reduced to

$$\int_0^\infty \lambda^n |R(\lambda)| \, d\lambda < \infty. \tag{7}$$

For a general ergodic potential $\{q_\omega(x)\}_{\omega\in\Omega}$ the non-negative quantity $\gamma(z)$ called Lyapounov exponent is crucial to investigate the spectrum of L_{q_ω}. This exponent is defined as

$$\gamma(z) = \lim_{x\to\infty} \frac{1}{x} \log \|U_\omega(x,z)\|,$$

where $U_\omega(x,z)$ is the 2×2 matrix solution to

$$\frac{d}{dx}U(x) = \begin{pmatrix} 0 & 1 \\ z - q_\omega(x) & 0 \end{pmatrix} U(x), \quad U(0) = I.$$

Due to the ergodicity it is known that this limit exists a.s. ω. *Floquet exponent* $w(z)$ which is an analog of the quantity in the case of periodic potentials is also defined by

$$w(z) = \mathbb{E}(m_{+,\omega}(z)),$$

and it is known that $\gamma(z) = -\operatorname{Re} w(z)$. Set

$$\chi(z) = \frac{\gamma(z)}{\operatorname{Im} z} - \operatorname{Im} w'(z).$$

Then, [Ko1] applied an identity

$$4\chi(z) = \mathbb{E}\left(|R(z)|^2 \left(\frac{1}{\operatorname{Im} m_+(z)} + \frac{1}{\operatorname{Im} m_-(z)}\right)\right) \tag{8}$$

to the study of the absolutely continuous spectrum of L_{q_ω}. Schwarz inequality together with (8) implies

$$\mathbb{E}(|R(z)|) \le \sqrt{2\chi(z)\operatorname{Im} w(z)}. \tag{9}$$

To apply (9) to the estimate (7) we have to use the conformal map ϕ again to shift the argument on the real axis to the one on the curve C_α. Consequently we have

Theorem 3 *$\mathcal{Q}$ contains the following potentials.*
(i) the Schwartz space $\mathcal{S}(\boldsymbol{R})$
(ii) ergodic potentials $\{q_\omega(x)\}_{\omega\in\Omega}$ satisfying

$$\int_0^\infty \lambda^n \gamma(\lambda) \, d\lambda < \infty$$

for any $n \ge 1$. This condition is satisfied if $\{q_\omega(x)\}_{\omega\in\Omega} \subset C_b^\infty(\boldsymbol{R})$.
(iii) any smooth potential q which coincides with an element of $\mathcal{S}(\boldsymbol{R})$ on a half axis and coincides with an ergodic potential satisfying the condition in (ii) on the other half axis.

Remark 1 *If we are interested only in the KdV equation, we have only to consider* $g_t(z) = e^{-4tz^3}$*, which relaxes the requirement that (6) should hold for all* $n \geq 1$ *to a requirement that (6) holds up to a certain fixed number* N*.*

Almost periodic functions in the Bohr's sense can be regarded as ergodic processes, hence Theorem 3 implies that the KdV equation with smooth almost periodic functions can be solved globally. However, the almost periodicity of the solution remains open to be proved.

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References

[BDGL] I. Binder, D. Damanik, M. Goldstein, M. Lukic: *Almost Periodicity in Time of Solutions of the KdV Equation*, arXiv:1509.07373

[CKSTT] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao: *Sharp global well-posedness for KdV and modified KdV on R and T*, Journal of the AMS, **16** (2003), 705-749

[DG] D. Damanik - M. Goldstein, *On the existence and uniqueness of global solutions of the KdV equation with quasiperiodic initial data*, J. Amer. Math. Soc., **29** (2016), 825-856

[Eg] I. E. Egorova, *The Cauchy problem for the KdV equation with almost periodic initial data whose spectrum is nowhere dense*, Adv. Soviet Math., **19** (1994), 181–208.

[GGKM] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura: *A method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. **19** (1967) 1095 - 1097.

[GS1] F. Gesztesy - B. Simon: *The xi function*, Acta Math., **176** (1996), 49-71

[GS2] F. Gesztesy - B. Simon: *A new approach to inverse spectral theory, II. General real potentials and the connection to the spectral measure*, Annals of Math., **152** (2000), 593-643

[GR] S. Grudsky - A. Rybkin: *Soliton Theory and Hankel Operators*, SIAM J. Math. Anal., 47(2015), 2283–2323.

[Ko1] S. Kotani: *Ljapounov indices determine absolutely continuous spectra of stationary random Schrödinger operators*, Stochastic Analysis (Katata/Kyoto, 1982), 225 - 247, North-Holland Math. Library, **32**, North-Holland, Amsterdam, 1984.

[Ko2] S. Kotani: *Determinantal formula of inverse spectral problem for Schrödinger operators and its application to KdV flow*: Proceedings of V International Conference Analysis and Mathematical Physics 19-23 June, 2017, Kharkiv, Ukraine

[KT] T. Kappeler, P. Topalov: *Global wellposedness of KdV in* $H^{-1}(\mathbb{T}, \mathbb{R})$: Duke Math. Journal 135(2006), 327-360.

[Mar] V. A. Marchenko: *The Cauchy problem for the KdV equation with non-decreasing initial data*, Springer Series in Nonlinear Dynamics, What is Integrability? ed. by V.E. Zakharov (1990), 273 - 318.

[Rem] C. Remling: *Generalized reflection coefficients*, Comm. Math. Phys. **337** (2015), 1011 - 1026

[Ryb1] A. Rybkin: *On the evolution of a reflection coefficient under the Kortweg-de Vries flow*, J. Math. Phys. **49** (2008), 15pp

[Ryb2] A. Rybkin: *The Hirota τ-function and well-posedness of the KdV equation with an arbitrary step like initial profile decaying on the right half line*, Nonlinearity **24** (2011), 2953-2990.

[Sat] M. Sato: *Soliton Equations as Dynamical Systems on an Infinite Dimensional Grassmann Manifolds*, Suriken Koukyuroku **439** (1981), 30 - 46. (http://www.kurims.kyoto-u.ac.jp/en/publi-01.html)

[SW] G. Segal - G. Wilson: *Loop groups and equations of KdV type*, Publ. IHES, **61** (1985), 5 - 65.

[Tsu] K. Tsugawa: *Local well-posedness of KdV equations with quasi-periodic initial data*, SIAM Journal of Mathematical Analysis, **44** (2012), 3412–3428.